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THE EFFECTIVE PROPERTIES OF PIEZOACTIVE MATRIX COMPOSITE MATERIALS

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Matrix composite materials consisting of a uniform component (the matrix), in which a large number of particles of filler (inclusions) are uniformly distributed, are considered. It is assumed that the components of these materials are ideally elastic and possess piezoelectric properties. One version of the self-consistent method (the effective-field method) is used to determine the effective electric and elastic characteristics of these materials, taking related electroelastic effects into account. As the first stage of the method, the problem of coupled electroelasticity for a homogeneous medium containing an isolated inhomogeneity is solved. The solution of this problem is found in analytic form for an ellipsoidal inclusion and a constant external field. The solution obtained is then used in a self-consistent scheme to construct an effective electroelastic operator of the composite, containing a random set of ellipsoidal inclusions. Explicit expressions are obtained for the electroelastic characteristics of composites, reinforced with spherical inclusions and continuous cylindrical fibres.

A systematic investigation of the electroelastic properties of different kinds of piezoactive composites using the conditional-averaging method can be found in a number of papers published by scientists of the Kiev School (see [1], where reference to original papers are given). The approach proposed in the present paper enables the detailed microstructure and interaction between the inclusions to be taken into account more completely.

1. Consider a uniform elastic piezoelectric material under isothermal conditions. The linear governing relations for such a material, which can be obtained by analysing the thermodynamic potentials (see, for example, [1–4]), have the form

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} - e_{ijk}E_k, \quad \frac{1}{4\pi}D_i = e^t_{ikl}\varepsilon_{kl} + \beta_{ik}E_k \tag{1.1}$$

Here σ and ε are the stress and strain tensors, E and D are the electric-field and induction vectors, respectively, $C = C^E$ is the elastic-moduli tensor for a fixed E vector, $\beta = \beta^\varepsilon$ is the permittivity tensor, e is the piezoelectric-constant tensor, which characterizes the related electroelastic effects, and the superscript t denotes the transposition operation.

Relations (1.1) can be conveniently written in the following short form

$$J = \mathbf{L}F, \quad J = \left\| \begin{matrix} \sigma \\ \frac{1}{4\pi}D \end{matrix} \right\|, \quad \mathbf{L} = \left\| \begin{matrix} C & -e \\ e^t & \beta \end{matrix} \right\|, \quad F = \left\| \begin{matrix} \varepsilon \\ E \end{matrix} \right\| \tag{1.2}$$

where the "matrix" L must be regarded as a linear operator which converts the tensor-vector pair [σ, D] into the analogous pair [ε, D] and which has symmetry of the electroelastic constants.

The relations inverse to (1.1), can be written in the form

$$F = \mathbf{M}J, \quad \mathbf{M} = \left\| \begin{matrix} S & d \\ -d^t & \eta \end{matrix} \right\| \tag{1.3}$$

$$S = S^D = (C + e\beta^{-1}e^t)^{-1}, \quad \eta = \eta^\sigma = (\beta + e^tC^{-1}e)^{-1}$$
$$d = Se\beta^{-1} = C^{-1}e\eta$$

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Since the basis of the effective-field method is the solution of the single-particle problem, we will first consider an unbounded piezoactive medium with electroelastic characteristics $\mathbf{L}^0$, containing a closed region V with a different electroelastic properties $\mathbf{L}$. We will start from the following combined system of equations of the theory of elasticity and electrical conductivity for a medium with an inhomogeneity

$$\nabla \mathbf{L} \nabla f(x) = 0, \quad f(x) = \begin{Vmatrix} u_i(x) \\ -\phi(x) \end{Vmatrix} \tag{1.4}$$
$$\mathbf{L}(x) = \mathbf{L}^0 + \mathbf{L}^1(x), \quad \mathbf{L}^1(x) = \mathbf{L}^1 V(x), \quad \mathbf{L}^1 = \mathbf{L} - \mathbf{L}^0, \quad \nabla_i = \partial/\partial x_i$$

where $u_i(x)$ are the components of the displacement vector, $\phi(x)$ is the electric potential at an arbitrary point x and $V(x)$ is the characteristic function of the region V. For our further consideration it will be more convenient to reduce the problem of determining the fields $u_i(x)$ and $\phi(x)$ to a system of integral equations, equivalent to the initial system of differential equations (1.4). This system has the form

$$F(x) = F^0(x) + \int_V \mathbf{P}(x - x')\mathbf{L}^1 F(x')dx', \quad x \in V$$
$$\mathbf{P}(x) = D\mathbf{G}(x)D, \quad D = \begin{Vmatrix} \text{def} & 0 \\ 0 & \text{grad} \end{Vmatrix} \tag{1.5}$$

Here $F^0(x)$ are the external elastic and electric fields which would arise in the main medium if there was no inhomogeneity and for specified conditions at infinity and $G(x)$ is Green's function of the combined system of equations of the theory of elasticity and electrical conductivity. For arbitrary anisotropy of the main medium this function is given by the expressions

$$\mathbf{G}(x) = \frac{1}{8\pi^2} \int_{|\xi|=1} \mathbf{G}(\xi)\delta(\xi x)dS_\xi \quad \mathbf{G}(\xi) = \begin{Vmatrix} G_{ij}(\xi) & \Gamma_i(\xi) \\ -\gamma_j(\xi) & g(\xi) \end{Vmatrix} \tag{1.6}$$
$$G_{ij} = \left(\Lambda_{ij} - \frac{1}{\lambda} H_i h_j\right)^{-1}, \quad \gamma_j = \frac{1}{\lambda} h_i G_{ij}, \quad g = -(\lambda + h_i \Lambda_{ij}^{-1} H_j)^{-1}$$
$$\Gamma_i = \Lambda_{ij}^{-1} H_j g, \quad \Lambda_{ij}(\xi) = C_{ikjl}^0 \xi_k \xi_l, \quad H_i(\xi) = e_{ikl}^0 \xi_k \xi_l$$
$$h_j(\xi) = e_{ljk}^{\prime} \xi_k \xi_l, \quad \lambda(\xi) = \beta_{ij} \xi_i \xi_j$$

When $x \in V$ the system of equations (1.5) defines the fields $\varepsilon(x)$ and $E(x)$ inside the inclusion, from which the field outside V can be established uniquely.

We will now assume that the inclusion has the form of an ellipsoid with semiaxes a_1, a_2 and a_3, which are specified by the relation $x_i(a^{-2})_{ij}x_j \leq 1$, $a_{ij} = a_i\delta_{ij}$ (there is no summation over i!). It can be shown that the integral operator with kernel $P(x)$ for an ellipsoidal region possesses the property of "polynomial conservatism" [5]. In particular, suppose the external fields are uniform in the region V (F^0 = const), and that this region itself is a sphere of radius a. If F = const, the problem reduces to evaluating the integral

$$\int_V \mathbf{P}(x - x')dx' = \frac{1}{8\pi^2} \int_{|\xi|=1} \mathbf{P}(\xi)dS_\xi \frac{\partial^2}{\partial p} \int_V \delta(p - \xi x')dx' \tag{1.7}$$
$$p = \xi \cdot x, \quad \mathbf{P}(\xi) = \xi \mathbf{G}(\xi)\xi$$

The integral over the region V is equal to the area of the circle which is formed by the section of a sphere by the plane $\xi \cdot x = p$, i.e. $\pi^2(a^2 - p^2)$, if $|p| \leq a$ and zero if $p > a$. When $x \in V$ the second derivative of this integral is equal to -2π and the right-hand side in (1.7) is constant.

A similar result is also obtained for an ellipsoid which is converted into the unit sphere using the coordinate transformation $t_i = a_{ij}^{-1}x_j$. In this case

$$\int_V \mathbf{P}(x - x')dx' = -\mathbf{P} = \text{const} \tag{1.8}$$

$$\mathbf{P} = \frac{|\det a|}{4\pi} \int\limits_{|\xi|=1} \mathbf{P}(\xi) \frac{dS_\xi}{\rho^3(\xi)}, \quad \rho(\xi) = \sqrt{\xi_i (a^2)_{ij} \xi_j}$$

Thus, for an external field $F^0(x)$, uniform in the region V, the integral equation (1.5) is converted into the algebraic equation

$$F = F^0 - \mathbf{P}\mathbf{L}^1 F \tag{1.9}$$

Solving this equation for F, we can express the strain field ε and the electric field strength E in terms of the external field ε^0 and E^0

$$F = \mathbf{A}F^0, \quad \mathbf{A} = (\mathbf{I} + \mathbf{P}\mathbf{L}^1)^{-1} \tag{1.10}$$

$$\mathbf{I} = \begin{Vmatrix} I_{ijkl} & 0 \\ 0 & \delta_{ik} \end{Vmatrix}, \quad I_{ijkl} = \delta_{i(k}\delta_{l)j}$$

2. We will now consider an unbounded elastic piezoelectric medium containing a random set of ellipsoidal inclusions, uniformly distributed in space, which occupy a system of isolated regions V_k with characteristic functions $V_k(x)$, $k = 1, 2, \ldots$. The system of equations for determining the strain fields $\varepsilon(x)$ and the electric field strength $E(x)$ in a medium with inhomogeneities has the following form, similar to (1.5)

$$F(x) = F^0(x) + \int \mathbf{P}(x - x')\mathbf{L}^1(x')V(x')F(x')dx' \tag{2.1}$$

Here $V(x)$ is the characteristic function of the region $V = \Sigma_k V_k$, occupied by the inclusions, and $\mathbf{L}^1(x)$ is a function which is identical with the constant quantity $\mathbf{L}(\omega_k)$ when $x \in V_k$ (ω_k is the set of geometrical parameters characterizing the orientation of the principal anisotropy axes of the kth inclusion).

To solve the problem of homogenization and to construct a macroscopic system of equations of the theory of coupled elasticity and electrical conductivity using the system of equations (2.1), we will employ the self-consistent scheme [6–8], the basic principle of which is as follows. We fix one of the typical samples of a random set of inclusions and we consider an arbitrary kth inclusion occupying a volume V_k. For this inclusion we introduce a local external field $F^*_{(k)}(x)$. This field is defined in Vk and is made up of the external field $F^0(x)$ and the perturbation fields of all the remaining inclusions.

We now introduce the field $F^*(x)$, which is identical with $F^*_{(k)}(x)$ when $x \in V_k$, and the function $V(x; x')$, defined as follows:

$$V(x; x') = \sum_{i \neq k} V_i(x'), \quad x \in V_k \tag{2.2}$$

This enables us to write, for an arbitrary point x in the region V

$$F^*(x) = F^0(x) + \int \mathbf{P}(x - x')\mathbf{L}^1(x')V(x; x')F(x')dx' \tag{2.3}$$

We will assume that the field $F^*(x)$ has the same structure in any of the regions occupied by the inclusions (hypothesis H_1 of the effective-field method). In particular, if we assume that this field is constant in each of the regions V_k but may be different for different inclusions), the field $F(x)$ ($x \in V$) is related to the local external field $F^*(x)$ by the relation obtained above when solving the single-particle problem for an ellipsoidal inhomogeneity

$$F(x) = \mathbf{A}(x)F^*(x) \tag{2.4}$$

Here $A(x)$ is a function which, when $x \in V_k$, is identical with the constant operator $\mathbf{A}(\omega_k)$ defined by (1.10).

Substituting (2.4) into the right-hand sides of Eqs (2.1) and (2.3) we can express the electroelastic fields at an arbitrary point of the medium in terms of the local external field

$$F(x) = F^*(x) + \int \mathbf{P}(x - x')\mathbf{L}^1(x')\mathbf{A}(x')F^*(x')V(x')dx' \tag{2.5}$$

and also obtain a self-consistent equation for determining this field

$$F^*(x) = F^0(x) + \int \mathbf{P}(x-x')\mathbf{L}^1(x')\mathbf{A}(x')F^*(x')V(x;x')dx' \tag{2.6}$$

If the set of inclusions is random, $F(x)$ and $F^*(x)$ are random functions. Averaging both sides of Eq. (2.5) over the ensemble of realizations of the random set of inclusions we can write

$$\langle F(x)\rangle = F^0(x) + \int \mathbf{P}(x-x')\langle \mathbf{L}^1(x')\mathbf{A}(x')F^*(x')V(x')|x'\rangle dx' \tag{2.7}$$

We have denoted the ensemble mean by the symbol $\langle\cdot|x'\rangle$ provided that the point x' is in the region V occupied by the inclusions.

We will now assume that the value of the random function $F^*(x)$ at points of the region V_i is statistically independent of the properties of the inclusion and of the geometrical characteristics of this region (hypothesis H_2 of the effective-field method). This enables us to represent the mean in the integrand in (2.7) in the form of the following product

$$\langle \mathbf{L}^1(x)\mathbf{A}(x)F^*(x)V(x)|x\rangle = \langle \mathbf{L}^1(x)\mathbf{A}(x)V(x)\rangle\langle F^*(x)|x\rangle \tag{2.8}$$

For a spatially homogeneous set of inclusions $\mathbf{L}^1(x)$ and $\mathbf{A}(x)$ are homogeneous random functions possessing ergodic properties. Using this property we obtain

$$\langle \mathbf{L}^1(x)\mathbf{A}(x)V(x)\rangle = n_0\langle v\mathbf{L}^A\rangle, \quad \mathbf{L}^A = \mathbf{L}^1\mathbf{A} \tag{2.9}$$

Here n_0 is the number density of the inclusions and v is the volume of a typical inclusion, while averaging on the right-hand side of (2.9) is assumed over the random dimensions and orientations of the ellipsoidal inhomogeneities.

The quantity $\langle F^*(x)|x\rangle$ is the ensemble average provided that the point x is in region V. This average will henceforth be called the effective field.

Taking (2.8) and (2.9) into account, Eq. (2.7) can be written in the form

$$\langle F(x)\rangle = F^0(x) + n_0\int \mathbf{P}(x-x')\langle v\mathbf{L}^A\rangle F'(x')dx' \tag{2.10}$$

Hence it follows that the mean field $\langle F(x)\rangle$ at an arbitrary point x of the composite material can be expressed in terms of the effective field $F'(x)$. Equation (2.6) is the starting equation for determining it. Averaging both sides of this equation with the condition $x \in V$, we can write

$$F'(x) = F^0(x) + \int \mathbf{P}(x-x')\langle \mathbf{L}^1(x')\mathbf{A}(x')F^*(x')V(x;x')|x'\rangle dx' \tag{2.11}$$

Hypothesis H_2 enables us to represent the mean in the integrand in this expression as follows:

$$\langle \mathbf{L}^1(x')\mathbf{A}(x')F^*(x')V(x;x')|x'\rangle = \langle \mathbf{L}^1(x')\mathbf{A}(x')V(x;x')|x'\rangle\langle F^*(x')|x';x\rangle \tag{2.12}$$

The symbol $\langle\cdot|x';x\rangle$ denotes the operation of averaging with the condition $x, x' \in V$. In general, the mean $\langle\cdot|x';x\rangle$ differs from $\langle\cdot|x\rangle$

Assuming that the properties of the inclusions are statistically independent of their position in space, the first factor on the right-hand side of (2.12) can be represented in the form

$$\langle \mathbf{L}^1(x')\mathbf{A}(x')V(x;x')|x'\rangle = n_0\langle v\mathbf{L}^A\rangle\Psi(x,x') \tag{2.13}$$

$$\Psi(x,x') = \langle V(x;x')|x\rangle/\langle V(x)\rangle$$

For a spatially homogeneous set of inclusions the function $\Psi(x,x')$ depends only on the difference of the arguments $\Psi(x,x') = \Psi(x-x')$. This function represents the distribution density of the inhomogeneities surrounding a typical inclusion, the centre of which is situated at the origin of coordinates. Sometimes we say that this function determines the form of the "correlation well", in which a typical inclusion in the composite is situated.

Equation (2.11) takes the form

$$F'(x) = F^0(x) + n_0\int \mathbf{P}(x-x')\langle v\mathbf{L}^A\rangle\Psi(x-x')\langle F^*(x')|x';x\rangle dx' \tag{2.14}$$

As already noted, the conditional average in the integrand in this expression differs from $F'(x)$. We can obtain an expression for this mean once again using Eq. (2.6), averaging both sides with the condition $x, x' \in V$. But then its right-hand side turns out to depend on a more complex conditional average. Repetition of this procedure leads to an infinite chain of related statistical equations in the conditional means of increasingly complex structure. Hence, the problem of closure arises, as it usually does in problems of this kind, which can only be solved approximately. In particular, we can close this chain at the first step by using the so-called "quasicrystalline approximation", proposed by Lax [9], by virtue of which the means $\langle\cdot\,|x';x\rangle$ and $\langle\cdot\,|x\rangle$ are identical. As a result we obtain

$$F'(x) = F^0(x) + n_0\int \mathbf{P}(x-x')\langle v\mathbf{L}^A\rangle\Psi(x-x')F'(x')dx' \tag{2.15}$$

Eliminating the external field $F^0(x)$ from (2.10) and (2.15), we arrive at an equation which relates the effective field $F'(x)$ to the mean field $\langle F(x)\rangle$ in the composite

$$F'(x) = \langle F(x)\rangle - n_0\int \mathbf{P}(x-x')\Phi(x-x')\langle v\mathbf{L}^A\rangle F'(x')dx' \tag{2.16}$$

$$\Phi(x) = 1 - \Psi(x)$$

If the set of inclusions possesses a certain symmetry (in the statistical sense), this affects the symmetry of the function $\Phi(x)$. In particular, if the set of inclusions is isotropic, this function will be spherically symmetrical, i.e. $\Phi(x) = \Phi(|x|)$.

Disturbance of the isotropy of the random set of inclusions may lead to the occurrence of a texture. We mean by texture here the difference in the symmetry of the tensors of the electroelastic characteristics of an inhomogeneous medium. In many important practical cases the symmetry of the texture can be described using a bivalent tensor b_{ij}, which defines the linear transformation of the space by which the function $\Phi(x)$ is converted into a spherically symmetric function

$$\Phi(b\cdot x) = \Phi(|x|) \tag{2.17}$$

Here the ellipsoid specified by the equation $(b\cdot x)^2 = 1$ will characterize the form of the correlation well. In general, of course, one cannot choose such a transformation.

For a random set of inclusions $\Phi(x)$ is a smooth function which rapidly approaches zero outside a region with dimensions of the order of the correlation well. If we neglect the change in the field $\langle F^*(x)\,|\,x\rangle$ in this region, Eq. (2.16) is converted into an algebraic equation

$$F'(x) = \langle F(x)\rangle - n_0\mathbf{\Pi}\langle v\mathbf{L}^A\rangle F'(x), \quad \mathbf{\Pi} = \int \mathbf{P}(x)\Phi(x)dx \tag{2.18}$$

Solving this equation for $F'(x)$ and substituting the result into the right-hand side of (2.10), we obtain

$$\langle F(x)\rangle = F^0(x) + n_0\int \mathbf{P}(x-x')\langle v\mathbf{L}^A\rangle D^0\langle F(x')\rangle dx' \tag{2.19}$$

$$D^0 = (I + n_0\Pi\langle v\mathbf{L}^A\rangle)^{-1}$$

Acting on both sides of this equation with the operator $\nabla\mathbf{L}^0$ and taking into account the relation

$$\nabla\mathbf{L}^0F^0(x) = 0, \quad \nabla\mathbf{L}^0\nabla\mathbf{G}(x) = -\mathbf{I}\delta(x)$$

we obtain that the mean elastic and electric fields in the composite material satisfy the equation

$$\nabla\mathbf{L}^*\langle F(x)\rangle = 0, \quad \mathbf{L}^* = \mathbf{L}^0 + n_0\langle v\mathbf{L}^A\rangle D^0 \tag{2.20}$$

which is identical in form with the equation of equilibrium of the theory of coupled electroelasticity for a certain homogeneous medium, the reaction of which to an external force is identical on average (macroscopically) with the reaction of a microinhomogeneous material. The quantity $\mathbf{L}^*$ in (2.20) is the operator of effective electroelastic characteristics of the piezoactive composite material.

3. We will consider some special cases.

The matrix in the composite material is isotropic but the inclusions are spheres of the same radius. In this case the operator **P** in (1.8) takes the form

$$\mathbf{P}=\left\| \begin{matrix} P_{ijkl} & 0 \\ 0 & p_{ik} \end{matrix} \right\| \tag{3.1}$$

$$P_{ijkl}=\frac{1}{9k_P}E^1_{ijkl}+\frac{1}{2\mu_P}E^2_{ijkl},\quad E^1_{ijkl}=\delta_{ij}\delta_{kl},\quad E^2_{ijkl}=I_{ijkl}-\frac{1}{3}E^1_{ijkl}$$

$$p_{ij}=\frac{1}{3\beta_0}\delta_{ij},\quad k_P=k_0+\frac{4}{3}\mu_0,\quad \mu_P=\frac{5\mu_0(3k_0+4\mu_0)}{6(k_0+2\mu_0)}$$

where k_0 and μ_0 are the bulk and shear moduli of the matrix.

We will assume that the electroelastic properties of the spherical inclusions are characterized by the cubic system of classes $\bar{4}3m$ and 23. In this case the tensors C, e and β can be represented in the form

$$C=k\mathbf{E}^1+2\mu\mathbf{E}^2+2(m-\mu)\mathbf{E}^3$$

$$E^3_{ijkl}=\sum_{r=1}^{3}\alpha_{ir}\alpha_{jr}\alpha_{kr}\alpha_{lr}-\frac{1}{3}E^1_{ijkl}$$

$$\beta_{ij}=\beta\delta_{ij},\quad \mathbf{e}=e\mathbf{U} \tag{3.2}$$

$$U_{ijk}=\alpha_{i1}\alpha_{j2}\alpha_{k3}+\alpha_{i2}\alpha_{j1}\alpha_{k3}+\alpha_{i3}\alpha_{j1}\alpha_{k2}+\alpha_{i1}\alpha_{j3}\alpha_{k2}+\alpha_{i2}\alpha_{j3}\alpha_{k1}+\alpha_{i3}\alpha_{j2}\alpha_{k1}$$

where α_{ir} is the conversion matrix from the crystal system of coordinates to the laboratory system of coordinates.

We will assume that the inclusions in the composite are distributed uniformly and isotropically in the matrix. Then, the correlation well has the form of a sphere and the operator $\mathbf{\Pi}$ in (2.18) is identical with the operator $\mathbf{P}$ defined by (3.1).

We will now consider two limiting cases.

1. Suppose the orientation of the principal axes of anisotropy of the inclusions is chaotic. Then, the composite as a whole is isotropic (there are no combined electroelastic effects in it) and is characterized by two effective elastic moduli k^* and μ^*, and also by a permittivity β^*. These quantities can be represented in the form

$$k^*=k_0+p\left(\frac{1}{k_A}-\frac{p}{k_P}\right)^{-1},\quad \mu^*=\mu_0+p\left(\frac{5}{2m_A+3\mu_A}-\frac{p}{\mu_P}\right)^{-1} \tag{3.3}$$

$$\beta^*=\beta_0+p\left(\frac{1}{\beta_A}-\frac{p}{3\beta_0}\right)^{-1}$$

where

$$k_A=k_1\left(1+\frac{k_1}{k_P}\right)^{-1},\quad \mu_A=\mu'\left(1+\frac{\mu'}{\mu_P}\right)^{-1},\quad m_A=m_1\left(1+\frac{m_1}{\mu_P}\right)^{-1} \tag{3.4}$$

$$\beta_A=\beta'\left(1+\frac{\beta'}{3\beta_0}\right)^{-1},\quad \mu'=\mu_1+\frac{e^2}{\beta_1+3\beta_0},\quad \beta'=\beta_1+\frac{e^2}{\mu_1+\mu_P}$$

$p=n_0\upsilon$ is the volume density of the inclusions and $k_1, \mu_1\dots$ here and henceforth denotes the difference between the corresponding characteristics of the inclusions and the matrix.

If the material of the inclusions is isotropic (i.e. $e=0$, $m=\mu$), formulae (3.3) reduce to the well-known expressions for the electroelastic constants of a composite containing a random set of isotropic spherical inclusions [6–8].

2. We will assume that the principal anisotropy axes of the inclusions are similarly oriented. In this case the composite as a whole possesses cubic symmetry of the same class as the inclusion. Its electroelastic properties are then characterized by the following effective elastic moduli

$$k^* = k_0 + p\left(\frac{1}{k_1} + \frac{1-p}{k_p}\right)^{-1}, \quad m^* = \mu_0 + p\left(\frac{1}{m_1} + \frac{1-p}{\mu_p}\right)^{-1}$$

$$\mu^* = \mu_0 + p\left[\frac{3\beta_0 + (1-p)\beta_1}{3\beta_0\mu_1 + (1-p)(\mu_1\beta_1 + e^2)} + \frac{1-p}{\mu_p}\right]^{-1} \tag{3.5}$$

its permittivity is

$$\beta^* = \beta_0 + p\left[\frac{\mu_p + (1-p)\mu_1}{\beta_1\mu_p + (1-p)(\beta_1\mu_1 + e^2)} + \frac{1-p}{3\beta_0}\right]^{-1} \tag{3.6}$$

and its piezoelastic constant is

$$e^* = 3p\beta_0\mu_p e[(\mu_p + (1-p)\mu_1)(3\beta_0 + (1-p)\beta_1) + (1-p)^2 e^2] \tag{3.7}$$

Note that taking the limit as $p \to 1$ formally in these formulae leads to physically non-contradictory results: $\mathbf{L}^* = \mathbf{L}$, although the effective-field hypotheses lose their meaning here.

A composite whose matrix is transversely isotropic. The tensors C^0, e^0 and β^0 for such a medium can be represented in the form

$$C^0 = k_0\mathbf{T}^2 + 2m_0\left(\mathbf{T}^1 - \frac{1}{2}\mathbf{T}^2\right) + l_0(\mathbf{T}^3 + \mathbf{T}^4) + 4\mu_0\mathbf{T}^5 + n_0\mathbf{T}^6$$

$$\mathbf{e} = e_1^0\mathbf{U}^1 + e_2^0\mathbf{U}^2 + e_3^0\mathbf{U}^3, \quad \beta = \beta_1^0\mathbf{t}^1 + \beta_2^0\mathbf{t}^2 \tag{3.8}$$

Here k_0, m_0, l_0, μ_0, n_0 are five independent elastic moduli of the transversely isotropic medium, e_1^0, e_2^0, e_3^0 are three piezoelastic constants and β_1^0, B_2^0 are two permittivities. The quantities $\mathbf{T}^i$, $\mathbf{U}^k$, $\mathbf{t}^t$ are the elements of the tensor bases, given by the expressions

$$T^1_{ijkl} = \theta_{i(k}\theta_{l)j}, \quad T^2_{ijkl} = \theta_{ij}\theta_{kl}, \quad T^3_{ijkl} = \theta_{ij}m_k m_l$$

$$T^4_{ijkl} = m_i m_j \theta_{kl}, \quad T^5_{ijkl} = \theta_{i)(k}m_{l)}m_{(j}, \quad T^6_{ijkl} = m_i m_j m_k m_l$$

$$U^1_{ijk} = \theta_{ij}m_k, \quad U^2_{ijk} = 2m_{(i}\theta_{j)k}, \quad U^3_{ijk} = m_i m_j m_k$$

$$t^1_{ij} = m_i m_j, \quad t^2_{ij} = \theta_{ij}, \quad \theta_{ij} = \delta_{ij} - m_i m_j$$

where m_i is the unit vector of the axis of symmetry of the material.

Suppose the inclusions in the composite have the form of continuous cylinders of the same radius similarly oriented parallel to the axis of symmetry of the properties of the matrix (a medium reinforced by unidirectional continuous fibres). To determine the operator $\mathbf{P}$ in this case we will use the general expression (1.8) and assume that the inclusion is a prolate spheroid ($a_1 = a_2 = a, a_3 > a$). We will change in (1.8) to a spherical system of coordinates ϕ, θ with polar axis directed along the axis of the spheroid. We make the change $\cos\theta = t$. We can then write

$$\mathbf{P} = \frac{1}{2\pi}\int_0^{2\pi} d\phi \int_{-1}^{1} \mathbf{P}(\phi, t)\psi(t, \delta)dt \tag{3.9}$$

$$\psi(t, \delta) = \frac{1}{2}\delta^2[\delta^2 + (1-\delta^2)t^2]^{-3/2}, \quad \delta = \frac{a}{a_3}$$

If we take the limit as $\delta \to 0$ this corresponds to an inclusion in the form of an infinite circular cylinder (a fibre) of radius a. Taking this limit in (3.9) we obtain

$$\mathbf{P} = \frac{1}{2\pi}\int_0^{2\pi} \mathbf{P}(\phi, 0)d\phi \tag{3.10}$$

To calculate the tensors occurring in this operator in explicit form we need to put $\xi_3 = 0$ in the tensors of $\mathbf{P}(\xi)$ in (1.8) (the vector ξ is then in the plane perpendicular to the fibre axis), substitute the expression obtained into (3.10) and evaluate the integrals.

For a transversely isotropic medium the components of the matrix $\mathbf{G}(\xi)$ in (1.6) when $\xi = (\xi_1, \xi_2, 0)$, have the form

$$\begin{gathered} G_{ij}(\xi) = -\frac{k_0}{m_0(k_0+m_0)}\xi_i\xi_j + \frac{1}{m_0}\theta_{ij} + \frac{1}{\mu_0'}m_i m_j \\ \Gamma_i = \gamma_i = \gamma m_i, \quad g = -\frac{1}{\beta_2'}, \quad \gamma = \frac{e_2^0}{\mu_0\beta_2^0 + (e_2^0)^2} \\ \mu_0' = \mu_0 + \frac{(e_2^0)^2}{\beta_2^0}, \quad \beta_2' = \beta_2^0 + \frac{(e_2^0)^2}{\mu_0} \end{gathered} \tag{3.11}$$

Substituting (3.11) into (3.10) and integrating over the unit circle, we obtain

$$\begin{gathered} \mathbf{P} = \left\| \begin{matrix} P & r \\ -r' & p \end{matrix} \right\| \\ P = P_1\mathbf{T}^2 + P_2\left(\mathbf{T}^1 - \frac{1}{2}\mathbf{T}^2\right) + \frac{1}{2\mu_0'}\mathbf{T}^5, \quad r = \frac{1}{4}\gamma\mathbf{U}^2 \\ p = \frac{1}{2\beta_2'}\mathbf{t}^2, \quad P_1 = \frac{1}{4(k_0+m_0)}, \quad P_2 = \frac{k_0+2m_0}{4m_0(k_0+m_0)} \end{gathered} \tag{3.12}$$

If we assume that the correlation well also has the form of a cylinder, parallel to the fibres, the general formula (2.20) will take the form

$$\mathbf{L}^* = \mathbf{L}^0 + p\mathbf{L}^1[\mathbf{I} + (1-p)\mathbf{P}\mathbf{L}^1]^{-1} \tag{3.13}$$

Suppose the fibre is also transversely isotropic with the axis of symmetry of the properties coinciding with their geometrical axis. The tensors of the electroelastic characteristics for these are defined by the same formulae (3.8), in which we must omit the zero subscript on the physical constants. As follows from (3.13), the composite as a whole will also be transversely isotropic and will be characterized by the following five effective elastic moduli

$$k^* = k_0 + pk_1 d(p), \quad m^* = m_0 + pm_1\left[1 + (1-p)\frac{m_1(k_0+2m_0)}{2m_0(k_0+m_0)}\right]$$

$$l^* = l_0 + pl_1 d(p), \quad \mu^* = \mu_0 + \frac{p}{\Delta(p)}\left[\mu_1 + \frac{(1-p)f}{2\beta_2'}\right]$$

$$n^* = n_0 + p\left[n_1 - \frac{(1-p)l_1^2 d(p)}{k_0+m_0}\right], \quad d(p) = \frac{k_0+m_0}{k_0+m_0+(1-p)k_1}$$

$$\Delta(p) = [1+(1-p)b][1+(1-p)B] - (1-p)^2 Qq, \quad f = \mu_1\beta_2^1 + (e_2^1)^2$$

$$b = \frac{1}{2}\left(\frac{\beta_2^1}{\beta_2'} + \gamma e_2^1\right), \quad B = \frac{1}{2}\left(\frac{\mu_1}{\mu_0'} + \gamma e_2^1\right)$$

$$q = \frac{1}{2}\left(\frac{e_2^1}{\beta_2'} - \gamma\mu_1\right), \quad Q = \frac{1}{2}\left(\gamma\beta_2^1 - \frac{e_2^1}{\mu_0'}\right)$$

three piezoelastic constants

$$e_1^* = e_1^0 + pe_1^1 d(p), \quad e_2^* = e_2^0 + \frac{p}{\Delta(p)}\left[e_2^1 + \frac{1}{2}(1-p)\gamma f\right]$$

$$e_3^* = e_3^0 + p\left[e_3^1 - \frac{(1-p)l_1 e_1^1 d(p)}{k_0 + m_0}\right]$$

and two permittivities

$$\beta_1^* = \beta_1^0 + p\left[\beta_1^1 + \frac{(1-p)(e_1^1)^2 d(p)}{k_0 + m_0}\right]$$

$$\beta_2^* = \beta_2^0 + \frac{p}{\Delta(p)}\left[\beta_2^1 + \frac{(1-p)f}{2\mu_0'}\right]$$

It can be seen from these formulae that taking into account the coupling of the elastic and electric fields affects only the values of the effective elastic modulus μ^*, the piezoelastic constants e_i^* $(i = 1, 2, 3)$ and the permittivities β_k^* $(k = 1, 2)$. As regards the elastic moduli of the composite k^*, m^*, l^*, n^*, they are defined by the same formulae [8] as in the case of purely elastic deformation.

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